

# SOLUTION OF THE CAUCHY PROBLEM FOR A TIME-DEPENDENT SCHRÖDINGER EQUATION

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**ABSTRACT.** We construct an explicit solution of the Cauchy initial value problem for the  $n$ -dimensional Schrödinger equation with certain time-dependent Hamiltonian operator of a modified oscillator. The dynamical  $SU(1,1)$  symmetry of the harmonic oscillator wave functions, Bargmann's functions for the discrete positive series of the irreducible representations of this group, the Fourier integral of a weighted product of the Meixner–Pollaczek polynomials, a Hankel-type integral transform and the hyperspherical harmonics are utilized in order to derive the corresponding Green function. It is then generalized to a case of the forced modified oscillator. The propagators for two models of the relativistic oscillator are also found. An expansion formula of a plane wave in terms of the hyperspherical harmonics and solution of certain infinite system of ordinary differential equations are derived as a by-product.

## 1. INTRODUCTION

The time-dependent Schrödinger equation for a free particle

$$i\psi_t + \Delta\psi = 0 \quad (1.1)$$

in the Euclidean space of  $n$  dimensions  $\mathbf{R}^n$  can be rewritten in a Hamiltonian form as

$$i\frac{\partial\psi}{\partial t} = H\psi, \quad H = -\Delta = \frac{1}{2} \sum_{s=1}^n (a_s + a_s^\dagger)^2, \quad (1.2)$$

where  $a_s^\dagger$  and  $a_s$  are the creation and annihilation operators, respectively, given by

$$a_s^\dagger = \frac{1}{i\sqrt{2}} \left( \frac{\partial}{\partial x_s} - x_s \right), \quad a_s = \frac{1}{i\sqrt{2}} \left( \frac{\partial}{\partial x_s} + x_s \right) \quad (1.3)$$

as in [26]. They satisfy the familiar commutation relations

$$[a_s, a_{s'}] = [a_s^\dagger, a_{s'}^\dagger] = 0, \quad [a_s, a_{s'}^\dagger] = \delta_{ss'} \quad (s, s' = 1, 2, \dots, n), \quad (1.4)$$

which are invariant under the transformation

$$a_s \rightarrow a_s(t) = e^{it} a_s, \quad a_s^\dagger \rightarrow a_s^\dagger(t) = a_s^\dagger e^{-it}. \quad (1.5)$$

The substitution

$$H \rightarrow H(t) = \frac{1}{2} \sum_{s=1}^n (a_s(t) + a_s^\dagger(t))^2 \quad (1.6)$$

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results in the time-dependent Schrödinger equation for a modified oscillator

$$i \frac{\partial \psi}{\partial t} = H(t) \psi \quad (1.7)$$

with the Hamiltonian of the form

$$H(t) = \frac{1}{2} \sum_{s=1}^n (a_s a_s^\dagger + a_s^\dagger a_s) + \frac{1}{2} e^{2it} \sum_{s=1}^n (a_s)^2 + \frac{1}{2} e^{-2it} \sum_{s=1}^n (a_s^\dagger)^2. \quad (1.8)$$

In the present paper we construct an exact solution of this equation subject to the initial condition

$$\psi(\mathbf{x}, t)|_{t=0} = \psi_0(\mathbf{x}), \quad (1.9)$$

where  $\psi_0(\mathbf{x})$  is an arbitrary square integrable complex valued function from  $\mathcal{L}^2(\mathbf{R}^n)$ . The explicit form of equation (1.7) is given by (6.1) below.

The paper is organized as follows. In section 2 we remind the reader about the solution of the stationary Schrödinger equation for the  $n$ -dimensional harmonic oscillator in hyperspherical coordinates and discuss the corresponding dynamical  $SU(1, 1)$  symmetry group. In section 3 we consider the eigenfunction expansion for the time-dependent Schrödinger equation (1.7). The series solution of the initial value problem (1.7)–(1.9) is obtained in section 6 after discussion of the Meixner–Pollaczek polynomials and evaluation of the Fourier integral of their weighted product in sections 4 and 5 respectively. Finally we construct the corresponding Green function in section 8, while introducing a required integral transform in section 7. A generalization to the case of the forced modified oscillator is given in section 9. An expansion formula for the plane wave in terms of the hyperspherical harmonics in  $\mathbf{R}^n$  and its special cases are discussed in section 10. A certain type of the time-dependent Schrödinger equation is considered in section 11 in an abstract form. The propagators for two models of the relativistic oscillator are derived in the next two sections. An axillary solution of an infinite system of the ordinary differential equations is found in section 14 as a by-product. Appendix at the end of the paper contains another required integral evaluation.

The exact solution of the one-dimensional time-dependent Schrödinger equation for a forced harmonic oscillator is constructed in [22], [23], [24], and [38]; see also references therein. These simple exactly solvable models may be of interest in a general treatment of the non-linear time-dependent Schrödinger equation; see [32], [35], [45], [52], [54], [63] and references therein. They may also be useful as test solutions for numerical methods of solving the time-dependent Schrödinger equation.

## 2. DYNAMICAL SYMMETRY OF THE HARMONIC OSCILLATOR IN $n$ -DIMENSIONS

Our time-dependent Hamiltonian operator (1.8) has the following structure

$$H(t) = H_0 + H_1(t), \quad (2.1)$$

where

$$H_0 = \frac{1}{2} \sum_{s=1}^n (a_s a_s^\dagger + a_s^\dagger a_s) \quad (2.2)$$

is the Hamiltonian of the  $n$ -dimensional harmonic oscillator and

$$H_1(t) = \frac{1}{2} e^{2it} \sum_{s=1}^n (a_s)^2 + \frac{1}{2} e^{-2it} \sum_{s=1}^n (a_s^\dagger)^2 \quad (2.3)$$

is the part depending on time  $t$ .

The stationary Schrödinger equation for the harmonic oscillator in  $n$ -dimensions

$$H_0 \Psi = E \Psi, \quad H_0 = \frac{1}{2} \sum_{s=1}^n \left( -\frac{\partial^2}{\partial x_s^2} + x_s^2 \right) \quad (2.4)$$

can be solved explicitly in Cartesian and (hyper)spherical coordinate systems; see, for example, [46], [55] and references therein.

In spherical coordinates  $r, \Omega$  given by a certain binary tree  $T$ , see [46], [55], [61] and references therein for a graphical approach of Vilenkin, Kuznetsov and Smorodinskiĭ to the theory of (hyper)spherical harmonics, we look for solution in the form

$$\psi = Y_{K\nu}(\Omega) R(r), \quad (2.5)$$

where  $Y_{K\nu}$  are the spherical harmonics constructed by the given tree  $T$ , the integer number  $K$  corresponds to the constant of separation of the variables at the root of  $T$  (denoted by  $K$  due to the tradition of the method of  $K$ -harmonics in nuclear physics [55]) and  $\nu = \{l_1, l_2, \dots, l_p\}$  is the set of all other subscripts corresponding to the remaining vertexes of the binary tree  $T$ . The radial wave function  $R(r)$ , which satisfies the normalization condition

$$\int_0^\infty R^2(r) r^{n-1} dr = 1, \quad (2.6)$$

is as follows

$$R = R_{NK}(r) = \sqrt{\frac{2[(N-K)/2]!}{\Gamma[(N+K+n)/2]}} \exp(-r^2/2) r^K L_{(N-K)/2}^{K+n/2-1}(r^2), \quad (2.7)$$

where  $L_k^\alpha(\xi)$  are the Laguerre polynomials; see [1], [2], [6], [19], [21], [28], [33], [46], [47], [60], [61], [64], and references therein for the advanced theory of the classical orthogonal polynomials.

The corresponding energy levels are

$$E = N + n/2, \quad (N - K)/2 = k = 0, 1, 2, \dots \quad (2.8)$$

and the normalized wave functions are given by

$$\Psi = \Psi_{NK\nu}(r, \Omega) = Y_{K\nu}(\Omega) R_{NK}(r), \quad (2.9)$$

where  $Y_{K\nu}(\Omega)$  are the spherical harmonics associated with the tree  $T$  and the radial functions  $R_{NK}(r)$  are defined by (2.7). The wave functions of the one-dimensional harmonic oscillator

$$\Psi_N(x) = \frac{1}{\sqrt{2^N N! \sqrt{\pi}}} e^{-x^2/2} H_N(x) \quad (2.10)$$

can be obtain from (2.9) by letting  $n = 1$  and  $K = 0, 1$  and invoking the familiar relations

$$H_{2k}(\xi) = (-1)^k 2^{2k} k! L_k^{-1/2}(\xi^2), \quad H_{2k+1}(\xi) = (-1)^k 2^{2k+1} k! \xi L_k^{1/2}(\xi^2) \quad (2.11)$$

between the Laguerre  $L_k^\alpha(\xi)$  and Hermite  $H_k(\xi)$  polynomials, respectively. Thus

$$\Psi_N(x) = \begin{cases} \frac{(-1)^{N/2}}{\sqrt{2}} R_{N0}(x), \\ \frac{(-1)^{(N-1)/2}}{\sqrt{2}} R_{N1}(x), \end{cases} \quad (2.12)$$

for even and odd  $N$ , respectively; see [46] and [55] for more details.

The  $n$ -dimensional oscillator wave functions (2.9) have the following group-theoretical properties. Introducing operators

$$\begin{aligned} J_+ &= \frac{1}{2} \sum_{s=1}^n (a_s^\dagger)^2, & J_- &= \frac{1}{2} \sum_{s=1}^n (a_s)^2, \\ J_0 &= \frac{1}{2} \sum_{s=1}^n (a_s^\dagger a_s + a_s a_s^\dagger) = \frac{1}{2} H_0, \end{aligned} \quad (2.13)$$

one can easily verify the following commutation relations

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = -2J_0. \quad (2.14)$$

For the Hermitian operators

$$J_x = \frac{1}{2} (J_+ + J_-), \quad J_y = \frac{1}{2i} (J_+ - J_-), \quad J_z = J_0, \quad (2.15)$$

we get

$$[J_x, J_y] = -iJ_z, \quad [J_y, J_z] = iJ_x, \quad [J_z, J_x] = iJ_y. \quad (2.16)$$

These commutation rules are valid for the infinitesimal operators of the non-compact group  $SU(1, 1)$ ; see, for example, [13], [25], [46] and [55] for more details.

We are going to use a different notation for the wave function (2.9) as follows

$$\psi_{jm}(\mathbf{x}) = \psi_{jm\{\nu\}}(\mathbf{x}) = \Psi_{NK\nu}(r, \Omega) = Y_{K\nu}(\Omega) R_{NK}(r), \quad (2.17)$$

where the new quantum numbers are  $j = K/2 + n/4 - 1$  and  $m = N/2 + n/4$  with  $m = j + 1, j + 2, \dots$ . The inequality  $m \geq j + 1$  holds because of the quantization rule (2.8), which gives  $N = K, K + 2, K + 4, \dots$ .

The operators  $J_\pm$  and  $J_0$  in the spherical coordinates  $r, \Omega$  have the form

$$J_\pm = \frac{1}{2} \left( H_0 - r^2 \pm \frac{n}{2} \pm r \frac{\partial}{\partial r} \right), \quad J_0 = \frac{1}{2} H_0 \quad (2.18)$$

and their actions on the oscillator wave functions are

$$J_\pm \psi_{jm} = \sqrt{(m \mp j)(m \pm j \pm 1)} \psi_{j, m \pm 1}, \quad J_0 \psi_{jm} = m \psi_{jm}, \quad (2.19)$$

whence

$$J^2 \psi_{jm} = j(j+1) \psi_{jm} \quad (2.20)$$

with  $J^2 = J_0^2 + J_0 - J_- J_+ = J_0^2 - J_0 - J_+ J_-$ . These relations coincide with the formulas that define the action of the infinitesimal operators  $J_\pm$  and  $J_0$  of the group  $SU(1, 1)$  on a basis  $|j, m\rangle$  of the irreducible representation  $\mathcal{D}_+^j$  belonging to the discrete positive series in an abstract Hilbert space [13]. In our realization of this basis in terms of the wave functions (2.17), depending on

the number  $n = \dim \mathbf{R}^n$ , the moment  $j = K/2 + n/4 - 1$  of the group  $SU(1, 1)$  and its projection  $m = N/2 + n/4$  may assume integer, half-integer and quarter-integer values. Thus the wave functions of the  $n$ -dimensional harmonic oscillator form a basis of the two-valued irreducible representation  $\mathcal{D}_+^j$  for the Lie algebra of  $SU(1, 1)$ .

Let us discuss in particular the group-theoretical properties of the wave functions (2.10) of the one-dimensional harmonic oscillator. In view of the definition (1.3) of the creation and annihilation operators,

$$ia\Psi_N = \sqrt{N}\Psi_{N-1}, \quad -ia^\dagger\Psi_N = \sqrt{N+1}\Psi_{N+1}, \quad (2.21)$$

where we have used the familiar differentiation formulas

$$H'_k(\xi) = 2kH_{k-1}(\xi) = 2\xi H_k(\xi) - H_{k+1}(\xi) \quad (2.22)$$

for the Hermite polynomials. In this case the relations (2.19) hold for the basis functions of the form

$$\psi_{jm} = \begin{cases} (-1)^{N/2} \Psi_N, & N = N^+ = 0, 2, 4, \dots \text{ for } j = -3/4, \\ (-1)^{(N-1)/2} \Psi_N, & N = N^- = 1, 3, 5, \dots \text{ for } j = -1/4, \end{cases} \quad (2.23)$$

where  $m = N/2 + 1/4$ . Thus the even and odd wave functions  $\Psi_N(x)$  of the one-dimensional harmonic oscillator form, respectively, bases for the two irreducible representations  $\mathcal{D}_+^j$  of the algebra  $SU(1, 1)$  with the moments  $j = -3/4$  for the even values of  $N = N^+$  and  $j = -1/4$  for odd  $N = N^-$ ; see [46] and [55] for more details.

### 3. EIGENFUNCTION EXPANSION FOR THE TIME-DEPENDENT SCHRÖDINGER EQUATION

In spirit of Dirac's time-dependent perturbation theory in quantum mechanics, see [20], [26], [30], [36], [41], [42], [53], we are looking for a solution of the initial values problem (1.7)–(1.9) as an infinite multiple series

$$\psi = \psi(\mathbf{x}, t) = \sum_{j\{\nu\}} \sum_{m=j+1}^{\infty} c_m(t) \psi_{jm\{\nu\}}(\mathbf{x}), \quad (3.1)$$

where  $\psi_{jm\{\nu\}}(\mathbf{x})$  are the oscillator wave functions (2.17) depending on the space coordinates  $\mathbf{x}$  only and  $c_m(t) = c_{jm\{\nu\}}(t)$  are yet unknown time-dependent coefficients.

The Hamiltonian (1.8) belongs to a more general type

$$H(t) = \omega J_0 + \delta(t) J_+ + \delta^*(t) J_- \quad (3.2)$$

with  $\delta(t) = e^{-i\omega t}$  and  $\omega = 2$ ; see (2.13) for the definition of operators  $J_\pm$  and  $J_0$ . It is convenient to proceed further with an arbitrary value of the parameter  $\omega$  and then to choose a particular value.

Substituting expansion (3.1) into the Schrödinger wave equation (1.7), with the help of (3.2), (2.19) and the orthogonality property of the oscillator wave functions (2.17), namely,

$$\int_{\mathbf{R}^n} \psi_{jm\{\nu\}}^*(\mathbf{x}) \psi_{j'm'\{\nu'\}}(\mathbf{x}) dv = \delta_{jj'} \delta_{mm'} (\delta_{\{\nu\}\{\nu'\}}), \quad (3.3)$$

we obtain an infinite system of the first order ordinary differential equations

$$i \frac{dc_m(t)}{dt} = \omega m c_m(t) + \delta(t) \sqrt{(m-j-1)(m+j)} c_{m-1}(t) \quad (3.4)$$

$$+ \delta^*(t) \sqrt{(m+j+1)(m-j)} c_{m+1}(t).$$

The following Ansatz

$$c_m(t) = \sqrt{\frac{(m-j-1)!}{(m+j)!}} e^{-i\omega m t} u_m(t) \quad (3.5)$$

results in

$$i \frac{du_m}{dt} = \delta(t) e^{i\omega t} (m+j) u_{m-1} + \delta^*(t) e^{-i\omega t} (m-j) u_{m+1}. \quad (3.6)$$

When  $\delta(t) = e^{-i\omega t}$  we obtain the system of the first order linear equations

$$i \frac{d\mathbf{u}}{dt} = \mathbf{A}\mathbf{u}, \quad \mathbf{u}(0) = \mathbf{u}^0, \quad (3.7)$$

or, explicitly,

$$i \frac{du_m}{dt} = (m+j) u_{m-1} + (m-j) u_{m+1} \quad (m = j+1, j+2, \dots, \infty) \quad (3.8)$$

with the Jacobi, or three diagonal, infinite matrix  $\mathbf{A}$  independent of time  $t$ . The system in hand should be solved subject to the initial conditions

$$\begin{aligned} u_m^0 &= u_m^0(0) = \sqrt{\frac{(m+j)!}{(m-j-1)!}} c_m(0) \\ &= \sqrt{\frac{(m+j)!}{(m-j-1)!}} \int_{\mathbf{R}^n} \psi_{jm\{\nu\}}^*(\mathbf{x}) \psi_0(\mathbf{x}) dv \end{aligned} \quad (3.9)$$

in view of (1.9), (3.1) and (3.5).

The solution of the initial value problem (3.8)–(3.9) can be constructed as follows

$$u_m(t) = \sum_{m'} u_{m'}^0 u_{mm'}(t), \quad (3.10)$$

where  $u_{mm'}(t)$  is a “Green” function, or particular solutions that satisfy the simplest initial conditions

$$u_{mm'}(0) = \delta_{mm'}. \quad (3.11)$$

Thus the solution of the original initial value problem (1.7)–(1.9) is given by

$$\psi(\mathbf{x}, t) = \sum_{j\{\nu\}} \sum_{m=j+1}^{\infty} c_m(t) \psi_{jm\{\nu\}}(\mathbf{x}) \quad (3.12)$$

with

$$\begin{aligned} c_m(t) &= \sqrt{\frac{(m-j-1)!}{(m+j)!}} e^{-i\omega m t} \\ &\times \sum_{m'=j+1}^{\infty} \sqrt{\frac{(m'+j)!}{(m'-j-1)!}} u_{mm'}(t) \int_{\mathbf{R}^n} \psi_{jm'\{\nu\}}^*(\mathbf{x}') \psi_0(\mathbf{x}') dv', \end{aligned} \quad (3.13)$$

where the Green function  $u_{mm'}(t)$  will be constructed in this paper in terms of the so-called Bargmann function [13], [46]; see also (6.8). It will be done in section 6 after discussion of some properties of the Meixner–Pollaczek polynomials and an integral evaluation in the next two sections.

#### 4. THE MEIXNER AND POLLACZEK POLYNOMIALS

The exact solution of the initial value problem (3.7) can be obtained with the help of the so-called Meixner–Pollaczek polynomials. They can be introduced in the following way.

The Meixner polynomials [39]–[40], [1], [19], [21], [46] are given by

$$y_n(x) = m_n^{(\gamma, \mu)}(x) = (\gamma)_n {}_2F_1 \left( \begin{matrix} -n, -x \\ \gamma \end{matrix} ; 1 - \frac{1}{\mu} \right), \quad (4.1)$$

where  $(\gamma)_n = \gamma(\gamma+1) \cdots (\gamma+n-1) = \Gamma(\gamma+n)/\Gamma(\gamma)$ ; see also [12] for the definition of the generalized hypergeometric series. Their orthogonality property is

$$\sum_{k=0}^{\infty} m_n^{(\gamma, \mu)}(k) m_l^{(\gamma, \mu)}(k) \frac{\mu^k (\gamma)_k}{k!} = \frac{n! (\gamma)_n}{\mu^n (1-\mu)^\gamma} \delta_{nl} \quad (4.2)$$

with  $\gamma > 0$  and  $0 < \mu < 1$ ; the proof is given, for example, in [46] and [47].

An important generating relation for the hypergeometric function

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{(r-k+1)_k}{k!} s^k {}_2F_1 \left( \begin{matrix} -k, -p \\ -r \end{matrix} ; u \right) {}_2F_1 \left( \begin{matrix} -k, -q \\ -r \end{matrix} ; v \right) \\ &= (1+s)^{r-p-q} (1+s-su)^p (1+s-sv)^q \\ & \quad \times {}_2F_1 \left( \begin{matrix} -p, -q \\ -r \end{matrix} ; -\frac{su v}{(1+s-su)(1+s-sv)} \right), \end{aligned} \quad (4.3)$$

which is due to Meixner [40], gives an extension of the orthogonality property as

$$\sum_{k=0}^{\infty} m_n^{(\gamma, \mu)}(k) m_l^{(\gamma, \mu)}(k) \frac{(\mu t)^k (\gamma)_k}{k!} \quad (4.4)$$

$$= (\gamma)_n (\gamma)_l \frac{(1-t)^{n+l}}{(1-\mu t)^{n+l+\gamma}} {}_2F_1 \left( \begin{matrix} -n, -l \\ \gamma \end{matrix} ; \frac{(1-\mu)^2 t}{\mu(1-t)^2} \right), \quad |t| < 1,$$

equation (4.2) arises in the limit  $t \rightarrow 1^-$ , and the explicit representation for the Poisson kernel

$$\sum_{n=0}^{\infty} m_n^{(\gamma, \mu)}(x) m_n^{(\gamma, \mu)}(y) \frac{(\mu t)^n}{(\gamma)_n n!} \quad (4.5)$$

$$= \frac{(1-t)^{x+y}}{(1-\mu t)^{x+y+\gamma}} {}_2F_1 \left( \begin{matrix} -x, -y \\ \gamma \end{matrix} ; \frac{(1-\mu)^2 t}{\mu(1-t)^2} \right), \quad |t| < 1.$$

The last two equations are related to each other in view of the self-duality

$$m_n^{(\gamma, \mu)}(k) / (\gamma)_n = m_k^{(\gamma, \mu)}(n) / (\gamma)_k \quad (n, k = 0, 1, 2, \dots) \quad (4.6)$$

of the Meixner polynomials; cf. (4.1).

The Meixner–Pollaczek polynomials [39], [48], [19], [8], [46] given by

$$p_n(x) = P_n^\lambda(x, \varphi) = \frac{e^{-in\varphi}}{n!} m_n^{(2\lambda, \mu)}(ix - \lambda), \quad \mu = e^{-2i\varphi} \quad (4.7)$$

with  $\lambda > 0$  and  $0 < \varphi < \pi$  satisfy the following three term recurrence relation

$$xP_n^\lambda(x, \varphi) = \frac{n+1}{2\sin\varphi} P_{n+1}^\lambda(x, \varphi) - (\lambda+n) \frac{\cos\varphi}{\sin\varphi} P_n^\lambda(x, \varphi) + \frac{2\lambda+n-1}{2\sin\varphi} P_{n-1}^\lambda(x, \varphi) \quad (4.8)$$

and the continuous orthogonality relation

$$\int_{-\infty}^{\infty} P_n^\lambda(x, \varphi) P_m^\lambda(x, \varphi) \rho(x) dx = \frac{\Gamma(2\lambda+n)}{n!} \delta_{nm} \quad (4.9)$$

with respect to the weight function

$$\rho(x) = \frac{1}{2\pi} (2\sin\varphi)^{2\lambda} |\Gamma(\lambda+ix)|^2 e^{(2\varphi-\pi)x}. \quad (4.10)$$

The Poisson kernel

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{n!}{(2\lambda)_n} t^n P_n^\lambda(x, \varphi) P_n^\lambda(y, \varphi) &= (1-t)^{-2\lambda} \left( \frac{1-t}{1-e^{-2i\varphi}t} \right)^{i(x+y)} \\ &\times {}_2F_1 \left( \begin{matrix} \lambda-ix, \lambda-iy \\ 2\lambda \end{matrix} ; -\frac{4t\sin^2\varphi}{(1-t)^2} \right), \quad |t| < 1 \end{aligned} \quad (4.11)$$

follows directly from (4.5) and (4.7). See [49] for a more general nonsymmetric form of this Poisson kernel; its  $q$ -expansions are given in [4] and [51]. An extension of the orthogonality property is the following Fourier integral

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-2ixt} P_n^\lambda(x, \varphi) P_m^\lambda(x, \varphi) e^{(2\varphi-\pi)x} |\Gamma(\lambda+ix)|^2 dx \\ = \frac{\Gamma(2\lambda+n) \Gamma(2\lambda+m)}{4^\lambda \Gamma(2\lambda) n! m!} \frac{e^{i\pi\lambda} (\sinh t)^{n+m}}{(\cos\varphi \sinh t + i \sin\varphi \cosh t)^{n+m+2\lambda}} \\ \times {}_2F_1 \left( \begin{matrix} -n, -m \\ 2\lambda \end{matrix} ; -\left( \frac{\sin\varphi}{\sinh t} \right)^2 \right). \end{aligned} \quad (4.12)$$

This integral evaluation will be given in the next section. In the limit  $t \rightarrow 0$  we obtain the orthogonality relation (4.9)–(4.10). See also [3], [5], [8], and [34] for the introduction and properties of the continuous Hahn polynomials that generalize the Meixner–Pollaczek polynomials.

## 5. EVALUATION OF THE INTEGRAL

We consider the analytic continuation of the relation (4.4) in the parameter  $\mu$ . By the Cauchy residue theorem the left-hand side of (4.4) can be rewritten as an integral over the contour  $C_1$  (see Figure 1), namely,

$$\begin{aligned} \frac{1}{2\pi i} \int_{C_1} m_n^{(\gamma, \mu)}(z) m_n^{(\gamma, \mu)}(z) t^z \tilde{\rho}(z) dz \\ = \frac{(1-t)^{x+y}}{(1-\mu t)^{x+y+\gamma}} {}_2F_1 \left( \begin{matrix} -x, -y \\ \gamma \end{matrix} ; \frac{(1-\mu)^2 t}{\mu(1-t)^2} \right), \end{aligned} \quad (5.1)$$



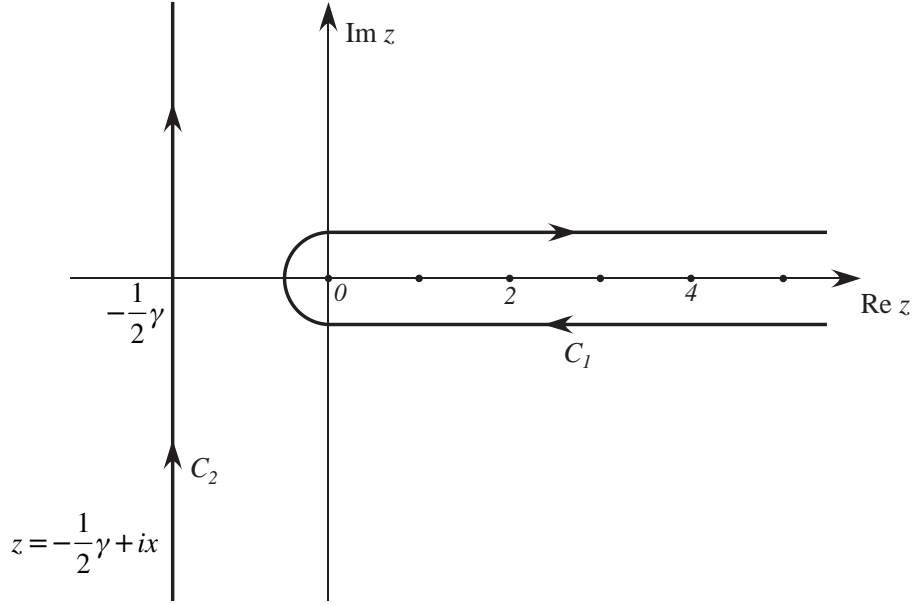


FIGURE 1. Contours in the complex plane.

where  $\tilde{\rho}(z) = (\gamma)_z \Gamma(-z) (-\mu)^z$  and  $|t| < 1$ . On the semicircle  $z = -\gamma/2 + R e^{i\theta}$  with  $-\pi/2 \leq \theta \leq \pi/2$  the following estimate

$$\tilde{\rho}(z) = O\left(R^{\gamma-1} \exp\left(R(\cos \theta \ln |\mu| - \sin \theta (\arg(-\mu) \pm \pi))\right)\right) \quad (5.2)$$

holds as  $R \rightarrow \infty$  [8]. Therefore for  $|\mu| < 1$  and  $|\arg(-\mu)| < \pi$  the contour  $C_1$  in (5.1) can be replaced by the contour  $C_2$  where  $z = -\gamma/2 + ix$  and  $-\infty < x < \infty$ . In view of the estimate (5.2), when  $|\arg(-\mu)| < \pi$  the integral in (5.1) converges uniformly on the contour  $C_2$ , where  $\theta = \pm\pi/2$ , for all values of  $|\mu|$ . As a result, this integral can be analytically continued in the parameter  $\mu$  to the entire complex  $\mu$ -plane with the cut along the positive real axis  $\text{Re } \mu > 0$ . In particular, equation (5.1) remains valid for both  $\mu = \exp(-2i\varphi)$  and  $z = -\gamma/2 + ix$  ( $\gamma > 0, 0 < \varphi < \pi$ ). The substitution (4.7) results in the integral (4.12) evaluation when  $t \rightarrow e^{-2t}$ .

## 6. SOLUTION OF THE INITIAL VALUE PROBLEM

We can now construct an explicit solution to the original Cauchy problem for the time-dependent Schrödinger equation of a modified oscillator in (1.7)–(1.9). More precisely, we will solve the following partial differential equation

$$\begin{aligned} i \frac{\partial \psi}{\partial t} = & \frac{1}{2} \sum_{s=1}^n \left( -(1 + \cos 2t) \frac{\partial^2 \psi}{\partial x_s^2} + (1 - \cos 2t) x_s^2 \psi \right) \\ & - \frac{i}{2} \sin 2t \sum_{s=1}^n \left( 2x_s \frac{\partial \psi}{\partial x_s} + \psi \right) \end{aligned} \quad (6.1)$$

subject to the initial condition

$$\psi(\mathbf{x}, t)|_{t=0} = \psi_0(\mathbf{x}), \quad \mathbf{x} \in \mathbf{R}^n \quad (6.2)$$

by using the eigenfunction expansion (3.12)–(3.13). Looking for a particular solution of the system (3.8) in the form

$$u_m(t) = e^{-2i\xi t} p_m(\xi), \quad (6.3)$$

where  $\xi$  is a spectral parameter, one gets

$$2\xi p_m(\xi) = (m+j)p_{m-1}(\xi) + (m-j)p_{m+1}(\xi) \quad (6.4)$$

which coincides with the recurrence relation for the Meixner–Pollaczek polynomials (4.8) when  $\lambda = j+1$  and  $\varphi = \pi/2$ . Thus

$$p_m(\xi) = P_{m-j-1}^{j+1}\left(\xi, \frac{\pi}{2}\right). \quad (6.5)$$

In view of the orthogonality relation (4.9), the Green function  $u_{mm'}(t)$ , or solution of the linear system (3.8) that satisfies the initial condition  $u_{mm'}(0) = \delta_{mm'}$ , can be obtain as the Fourier integral over the spectral parameter

$$\begin{aligned} u_{mm'}(t) &= \frac{1}{d_{m'}^2} \int_{-\infty}^{\infty} e^{-2i\xi t} p_m(\xi) p_{m'}(\xi) \rho(\xi) d\xi \\ &= \frac{(m'-j-1)! 2^{2j+2}}{(m'+j)! 2\pi} \\ &\quad \times \int_{-\infty}^{\infty} e^{-2i\xi t} P_{m-j-1}^{j+1}\left(\xi, \frac{\pi}{2}\right) P_{m'-j-1}^{j+1}\left(\xi, \frac{\pi}{2}\right) |\Gamma(j+1+i\xi)|^2 d\xi \\ &= \frac{(m+j)!}{(m-j-1)!} \frac{(-i)^{(m-j-1)+(m'-j-1)}}{\Gamma(2j+2)} \frac{(\sinh t)^{m+m'-2j-2}}{(\cosh t)^{m+m'}} \\ &\quad \times {}_2F_1\left(\begin{matrix} -m+j+1, -m'+j+1 \\ 2j+2 \end{matrix}; -\frac{1}{\sinh^2 t}\right), \end{aligned} \quad (6.6)$$

where the last integral has been evaluated with the help of the integral representation (4.12).

The generalized spherical harmonics for the discrete positive series  $\mathcal{D}_+^j$  of the non-compact Lorentz group  $SU(1, 1)$  are [13], [46]

$$T_{mm'}^j(\alpha, \tau, \gamma) = e^{-im\alpha} v_{mm'}^j(\tau) e^{-im'\gamma}, \quad (6.7)$$

where the Bargmann functions  $v_{mm'}^j(\tau)$  are given by

$$\begin{aligned} v_{mm'}^j(\tau) &= \langle jm | e^{-i\tau J_y} | jm' \rangle = e^{-n\tau/4} \int_0^\infty R_{NK}(r) R_{NK'}(e^{-\tau/2}r) r^{n-1} dr \\ &= \frac{(-1)^{m-j-1}}{\Gamma(2j+2)} \sqrt{\frac{(m+j)!(m'+j)!}{(m-j-1)!(m'-j-1)!}} \left(\sinh \frac{\tau}{2}\right)^{-2j-2} \left(\tanh \frac{\tau}{2}\right)^{m+m'} \\ &\quad \times {}_2F_1\left(\begin{matrix} -m+j+1, -m'+j+1 \\ 2j+2 \end{matrix}; -\frac{1}{\sinh^2(\tau/2)}\right). \end{aligned} \quad (6.8)$$

This implies the symmetry relation

$$v_{mm'}^j(\tau) = (-1)^{m-m'} v_{m'm}^j(\tau) \quad (6.9)$$

and the differentiation formula

$$2\frac{d}{d\tau} v_{m'm}^j(\tau) = \sqrt{(m-j-1)(m+j)} v_{m',m-1}^j(\tau) \quad (6.10)$$

$$- \sqrt{(m+j+1)(m-j)} v_{m',m+1}^j(\tau),$$

which follows directly from (6.8) and (2.19). Also

$$\sum_{m''=j+1}^{\infty} v_{mm''}^j(\tau) v_{m'm''}^j(\tau) = \delta_{mm'}, \quad (6.11)$$

see [13], [46], and [56] for more details.

Combining equations (3.12)–(3.13), (6.6) and (6.8)–(6.9) together, we finally arrive at the following expression in terms of the Bargmann functions

$$c_m(t) = e^{-i\omega m t} \sum_{m'=j+1}^{\infty} i^{m'-m} v_{m'm}^j(2t) \int_{\mathbf{R}^n} \psi_{jm'\{\nu\}}^*(\mathbf{x}') \psi_0(\mathbf{x}') dv' \quad (6.12)$$

for the time-depending coefficients  $c_m(t)$  in the expansion (3.12), namely,

$$\psi(\mathbf{x}, t) = \sum_{j\{\nu\}} \sum_{m=j+1}^{\infty} c_m(t) \psi_{jm\{\nu\}}(\mathbf{x}), \quad (6.13)$$

for the solution of the original initial value problem (1.7)–(1.9) with  $\omega = 2$ . With the help of the differentiation formula (6.10) one can easily verify that these coefficients  $c_m(t)$  satisfy the system of ordinary differential equations (3.4) with corresponding initial conditions. This gives a direct proof that our solution  $\psi(\mathbf{x}, t)$  does satisfy the initial value problem (1.7)–(1.9).

Thus constructed solution  $\psi(\mathbf{x}, t)$  belongs to the space of the square integrable functions  $\mathcal{L}^2(\mathbf{R}^n)$  for all times provided that the initial function  $\psi_0(\mathbf{x})$  is of the same class  $\psi_0 \in \mathcal{L}^2(\mathbf{R}^n)$ . Indeed, in this case the condition

$$\sum_{m=j+1}^{\infty} |c_m(t)|^2 = 1 \quad (6.14)$$

holds for all values of  $t$  in view of the unitary relation (6.11) of Bargmann's functions. We shall take advantage of the group-theoretical meaning of this solution in order to construct the corresponding Green function later.

## 7. THE FINITE “ROTATION” OPERATORS

We apply a general approach to a certain type of the integral transforms [4], [51], [59], [58], which was originated by Wiener [62], to our problem. Consider the following bilinear sum

$$\begin{aligned} S_t(r, r') &= \sum_{N=K, K+2, \dots, \infty} R_{NK}(r) R_{NK}(r') t^{(N-K)/2} \\ &= 2 \exp\left(-\left(r^2 + r'^2\right)/2\right) (rr')^K \\ &\quad \times \sum_{k=0}^{\infty} \frac{k!}{\Gamma(K + n/2 + k)} L_k^{K+n/2-1}(r^2) L_k^{K+n/2-1}(r'^2) t^k \end{aligned} \quad (7.1)$$

for the oscillator radial functions (2.7). With the help of the Poisson kernel for the Laguerre polynomials

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{k!}{(\alpha+1)_k} L_k^\alpha(x) L_k^\alpha(y) t^k \\ = (1-t)^{-\alpha-1} \exp\left(-\frac{(x+y)t}{1-t}\right) {}_0F_1\left(\begin{matrix} - \\ \alpha+1 \end{matrix}; \frac{xyt}{(1-t)^2}\right), \end{aligned} \quad (7.2)$$

see, for example, [50], p. 212; we derive the following closed form

$$\begin{aligned} S_t(r, r') &= \frac{2}{\Gamma(K+n/2)} (rr')^K (1-t)^{-(K+n/2)} \exp\left(-\frac{r^2+r'^2}{2} \frac{1+t}{1-t}\right) \\ &\times {}_0F_1\left(\begin{matrix} - \\ K+n/2 \end{matrix}; \frac{(rr')^2 t}{(1-t)^2}\right), \quad |t| \leq 1, \quad t \neq 1 \end{aligned} \quad (7.3)$$

for this kernel. In the case  $t = e^{i\alpha}$  one gets

$$\begin{aligned} S_{e^{i\alpha}}(r, r') &= \frac{2^{-2j-1} e^{i(\pi-\alpha)(j+1)}}{\Gamma(2j+2) (\sin(\alpha/2))^{2j+2}} (rr')^{2j+2-n/2} \exp\left(\frac{r^2+r'^2}{2i \tan(\alpha/2)}\right) \\ &\times {}_0F_1\left(\begin{matrix} - \\ 2j+2 \end{matrix}; -\frac{(rr')^2}{4 \sin^2(\alpha/2)}\right), \end{aligned} \quad (7.4)$$

where we use the  $SU(1,1)$  moment  $j = K/2 + n/4 - 1$ .

Using the orthogonality property of the radial functions

$$\int_0^\infty R_{NK}(r) R_{N'K}(r) r^{n-1} dr = \delta_{NN'}, \quad (7.5)$$

from (7.1) we obtain the following integral equation

$$t^{(N-K)/2} R_{NK}(r) = \int_0^\infty S_t(r, r') R_{NK}(r') (r')^{n-1} dr', \quad |t| < 1. \quad (7.6)$$

In the  $SU(1,1)$  notations for the oscillator wave functions(2.17), when  $t = e^{i\alpha}$ , it takes the form

$$e^{im\alpha} \psi_{jm\{\nu\}}(\mathbf{x}) = \int_0^\infty G_\alpha^j(r, r') \psi_{jm\{\nu\}}(\mathbf{x}') (r')^{n-1} dr', \quad (7.7)$$

where

$$\begin{aligned} G_\alpha^j(r, r') &= \frac{2^{-2j-1} e^{i\pi(j+1)}}{\Gamma(2j+2) (\sin(\alpha/2))^{2j+2}} (rr')^{2j+2-n/2} \exp\left(\frac{r^2+r'^2}{2i \tan(\alpha/2)}\right) \\ &\times {}_0F_1\left(\begin{matrix} - \\ 2j+2 \end{matrix}; -\frac{(rr')^2}{4 \sin^2(\alpha/2)}\right) \end{aligned} \quad (7.8)$$

and  $\mathbf{x}/r = \mathbf{x}'/r' = \mathbf{n} = \mathbf{n}(\Omega)$  with  $\mathbf{n}^2 = 1$ . The following symmetry properties hold

$$G_\alpha^j(r, r') = G_\alpha^j(r', r) = (G_{-\alpha}^j(r, r'))^* \quad (7.9)$$

and the formal orthogonality relation is

$$\int_{r''=0}^{\infty} (G_{\alpha}^j(r, r''))^* G_{\alpha}^j(r'', r') (r'')^{n-1} dr'' = \frac{\delta(r - r')}{r^{n-1}}, \quad (7.10)$$

where  $\delta(r)$  is the Dirac delta function. Also,

$$\sum_{m=j+1}^{\infty} e^{im\alpha} \psi_{jm\{\nu\}}(\mathbf{x}) \psi_{jm\{\nu\}}^*(\mathbf{x}') = Y_{K\nu}(\Omega) Y_{K\nu}(\Omega') G_{\alpha}^j(r, r'), \quad \alpha \neq 0. \quad (7.11)$$

In particular, when  $\alpha = \pm\pi/2$ , one gets

$$(\pm i)^m \psi_{jm\{\nu\}}(\mathbf{x}) = \int_0^{\infty} G_{\pm\pi/2}^j(r, r') \psi_{jm\{\nu\}}(\mathbf{x}') (r')^{n-1} dr' \quad (7.12)$$

with

$$G_{\pm\pi/2}^j(r, r') = \frac{e^{\pm i\pi(j+1)} (rr')^{2j+2-n/2}}{2^j \Gamma(2j+2)} e^{\pm(r^2+r'^2)/2i} {}_0F_1\left(\begin{matrix} - \\ 2j+2 \end{matrix}; -\frac{1}{2}(rr')^2\right) \quad (7.13)$$

These formulas will allow us to find a different form of the solution (6.12)–(6.13) in the next section.

In the process we found out that the finite “rotation” operator  $e^{-i\alpha J_z}$  of the group  $SU(1, 1)$  acts on the oscillator wave functions (2.17) as the following integral operator

$$\begin{aligned} e^{-i\alpha J_z} \psi_{jm\{\nu\}}(\mathbf{x}) &= \int_0^{\infty} G_{-\alpha}^j(r, r') \psi_{jm\{\nu\}}(\mathbf{x}') (r')^{n-1} dr' \\ &= Y_{K\nu}(\Omega) \int_0^{\infty} G_{-\alpha}^j(r, r') R_{NK}(r') (r')^{n-1} dr' \end{aligned} \quad (7.14)$$

with the kernel explicitly given by (7.8). Also, in view of (6.8),

$$\begin{aligned} e^{-i\tau J_y} \psi_{jm\{\nu\}}(\mathbf{x}) &= \sum_{m'=j+1}^{\infty} v_{m'm}^j(\tau) \psi_{jm'\{\nu\}}(\mathbf{x}) \\ &= e^{-n\tau/4} \psi_{jm\{\nu\}}(e^{-\tau/2} \mathbf{x}) = e^{-n\tau/4} Y_{K\nu}(\Omega) R_{NK}(e^{-\tau/2} r), \end{aligned} \quad (7.15)$$

see [46] for more details.

## 8. AN INTEGRAL FORM OF THE SOLUTION

We rewrite the coefficients (6.12) in the form

$$c_m(t) = e^{-im(\omega t + \pi/2)} \sum_{m'=j+1}^{\infty} v_{m'm}^j(2t) \int_{\mathbf{R}^n} \left( (-i)^{m'} \psi_{jm'\{\nu\}}(\mathbf{x}') \right)^* \psi_0(\mathbf{x}') dv' \quad (8.1)$$

and use the integral transform (7.12) as

$$(-i)^{m'} \psi_{jm'\{\nu\}}(\mathbf{x}') = \int_0^{\infty} G_{-\pi/2}^j(r', r'') \psi_{jm'\{\nu\}}(\mathbf{x}'') (r'')^{n-1} dr'', \quad (8.2)$$

where  $\mathbf{x}'/r' = \mathbf{x}''/r'' = \mathbf{n}' = \mathbf{n}(\Omega')$ , in order to obtain

$$c_m(t) = e^{-im(\omega t + \pi/2)} \quad (8.3)$$

$$\begin{aligned}
& \times \int_{\mathbf{R}^n} \left( \int_0^\infty G_{-\pi/2}^j(r', r'') \left( \sum_{m'=j+1}^\infty v_{m'm}^j(2t) \psi_{jm'\{\nu\}}(\mathbf{x}'') \right) (r'')^{n-1} dr'' \right)^* \psi_0(\mathbf{x}') dv' \\
& = e^{-im(\omega t + \pi/2)} \int_{\mathbf{R}^n} \left( \int_0^\infty G_{-\pi/2}^j(r', r'') (e^{-2itJ_y} \psi_{jm\{\nu\}}(\mathbf{x}'')) (r'')^{n-1} dr'' \right)^* \psi_0(\mathbf{x}') dv' \\
& = e^{-im(\omega t + \pi/2) - nt/2} \int_{\mathbf{R}^n} \left( \int_0^\infty G_{\pi/2}^j(r', r'') \psi_{jm\{\nu\}}^*(e^{-t}\mathbf{x}'') (r'')^{n-1} dr'' \right) \psi_0(\mathbf{x}') dv'
\end{aligned}$$

with the help of (7.15). Substitution into the eigenfunction expansion (6.13) gives

$$\begin{aligned}
\psi(\mathbf{x}, t) &= e^{-nt/2} \int_{\mathbf{R}^n} \psi_0(\mathbf{x}') \\
&\times \left( \sum_{j\{\nu\}} \int_0^\infty G_{\pi/2}^j(r', r'') \left( \sum_{m=j+1}^\infty e^{-im(\omega t + \pi/2)} \psi_{jm\{\nu\}}(\mathbf{x}) \psi_{jm\{\nu\}}^*(e^{-t}\mathbf{x}'') \right) (r'')^{n-1} dr'' \right) dv',
\end{aligned} \tag{8.4}$$

where by (2.17) and (7.11)

$$\sum_{m=j+1}^\infty e^{-im(\omega t + \pi/2)} \psi_{jm\{\nu\}}(\mathbf{x}) \psi_{jm\{\nu\}}^*(e^{-t}\mathbf{x}'') = Y_{K\nu}(\Omega) Y_{K\nu}^*(\Omega') G_{-\omega t - \pi/2}^j(r, e^{-t}r''), \tag{8.5}$$

and our solution takes the form

$$\begin{aligned}
\psi(\mathbf{x}, t) &= e^{-nt/2} \int_{\mathbf{R}^n} \psi_0(\mathbf{x}') \\
&\times \left( \sum_{j\{\nu\}} Y_{K\nu}(\Omega) Y_{K\nu}^*(\Omega') \int_0^\infty G_{\pi/2}^j(r', r'') G_{-\omega t - \pi/2}^j(r, e^{-t}r'') (r'')^{n-1} dr'' \right) dv'
\end{aligned} \tag{8.6}$$

with  $\omega = 2$ . The last integral can be evaluated with the help of the formula (15.5) from the appendix at the end of the paper

$$\begin{aligned}
& \int_0^\infty G_{\pi/2}^j(r', r'') \left( G_{2t + \pi/2}^j(r, e^{-t}r'') \right)^* (r'')^{n-1} dr'' \\
&= e^{nt/2} \frac{e^{i\pi(j+1)}}{2^j \Gamma(2j+1)} \frac{(rr')^{2j+2-n/2}}{(e^{-t} \cos(t + \pi/4) - e^t \sin(t + \pi/4))^{2j+2}} \\
&\times \exp \left( \frac{r^2 e^t \cos(t + \pi/4) + e^{-t} \sin(t + \pi/4)}{2i e^{-t} \cos(t + \pi/4) - e^t \sin(t + \pi/4)} \right) \\
&\times \exp \left( \frac{(r')^2 e^{-t} \cos(t + \pi/4) + e^t \sin(t + \pi/4)}{2i e^{-t} \cos(t + \pi/4) - e^t \sin(t + \pi/4)} \right) \\
&\times {}_0F_1 \left( - ; -\frac{(rr')^2}{2(e^{-t} \cos(t + \pi/4) - e^t \sin(t + \pi/4))^2} \right).
\end{aligned} \tag{8.7}$$

As a result, the solution of the Cauchy initial value problem (1.7)–(1.9) is given by

$$\psi(\mathbf{x}, t) = \int_{\mathbf{R}^n} G_t(\mathbf{x}, \mathbf{x}') \psi_0(\mathbf{x}') dv', \tag{8.8}$$

where the Green function is

$$G_t(\mathbf{x}, \mathbf{x}') = \sum_{K\nu} Y_{K\nu}(\Omega) Y_{K\nu}^*(\Omega') \mathcal{G}_t^K(r, r') \quad (8.9)$$

with

$$\begin{aligned} \mathcal{G}_t^K(r, r') &= \frac{e^{-i\pi(2K+n)/4}}{2^{K+n/2-1}\Gamma(K+n/2)} \frac{(rr')^K}{(\cos t \sinh t + \sin t \cosh t)^{K+n/2}} \\ &\times \exp\left(i \frac{(r^2 + (r')^2) \cos t \cosh t - (r^2 - (r')^2) \sin t \sinh t}{2(\cos t \sinh t + \sin t \cosh t)}\right) \\ &\times {}_0F_1\left(\begin{matrix} - \\ K+n/2 \end{matrix}; -\frac{(rr')^2}{4(\cos t \sinh t + \sin t \cosh t)^2}\right). \end{aligned} \quad (8.10)$$

The details of the calculations are left to the reader.

The Green function can also be independently found by separation of the variables in the Cartesian coordinates. Indeed, when  $n = 1$  and  $K = 0, 1$  with the help of the familiar relations

$$\cos \alpha = {}_0F_1\left(\begin{matrix} - \\ 1/2 \end{matrix}; -\frac{\alpha^2}{4}\right), \quad \sin \alpha = \alpha {}_0F_1\left(\begin{matrix} - \\ 3/2 \end{matrix}; -\frac{\alpha^2}{4}\right) \quad (8.11)$$

our equations (8.9)–(8.10) can be reduced to

$$\begin{aligned} G_t(x, x') &= \frac{1}{2} (\mathcal{G}_t^0(x, x') + \mathcal{G}_t^1(x, x')) \\ &= \frac{1}{\sqrt{2\pi i (\cos t \sinh t + \sin t \cosh t)}} \\ &\times \exp\left(\frac{(x^2 - (x')^2) \sin t \sinh t + 2xx' - (x^2 + (x')^2) \cos t \cosh t}{2i (\cos t \sinh t + \sin t \cosh t)}\right), \end{aligned} \quad (8.12)$$

which gives the Green function for the one-dimensional Schrödinger equation (6.1); we shall elaborate on the one-dimensional case in the next section. Thus, in the general case,

$$\begin{aligned} G_t(\mathbf{x}, \mathbf{x}') &= \prod_{s=1}^n G_t(x_s, x'_s) \\ &= \left(\frac{1}{2\pi i (\cos t \sinh t + \sin t \cosh t)}\right)^{n/2} \\ &\times \exp\left(\frac{(\mathbf{x}^2 - \mathbf{x}'^2) \sin t \sinh t + 2\mathbf{x} \cdot \mathbf{x}' - (\mathbf{x}^2 + \mathbf{x}'^2) \cos t \cosh t}{2i (\cos t \sinh t + \sin t \cosh t)}\right), \end{aligned} \quad (8.13)$$

and equation (8.9) gives an expansion formula for this Green function in terms of the corresponding hyperspherical harmonics. This type of oscillatory integrals is discussed in [57].

The time evolution operator for the time-dependent Schrödinger equation (1.7) can formally be written as

$$U(t, t_0) = \mathbf{T} \left( \exp \left( -\frac{i}{\hbar} \int_{t_0}^t H(t') dt' \right) \right), \quad (8.14)$$

where  $\mathbf{T}$  is the time ordering operator which orders operators with larger times to the left [18], [26]. Namely, this unitary operator takes a state at time  $t_0$  to a state at time  $t$ , so that

$$\psi(x, t) = U(t, t_0) \psi(x, t_0) \quad (8.15)$$

and

$$U(t, t_0) = U(t, t') U(t', t_0), \quad (8.16)$$

$$U^{-1}(t, t_0) = U^\dagger(t, t_0) = U(t_0, t). \quad (8.17)$$

We have constructed this time evolution operator explicitly in (8.8), as the integral operator with the kernel given by (8.13), for the particular form of the time-dependent Hamiltonian operator of a modified oscillator in (1.8).

## 9. THE FORCED MODIFIED OSCILLATOR

In the previous section, among other things, we have solved the following one-dimensional time-depending Schrödinger equation for a modified oscillator

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2} (1 + \cos 2t) \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2} (1 - \cos 2t) x^2 \psi - \frac{i}{2} \sin 2t \left( 2x \frac{\partial \psi}{\partial x} + \psi \right) \quad (9.1)$$

on the infinite interval subject to the initial condition

$$\psi(x, t)|_{t=0} = \psi_0(x) \quad (-\infty < x < \infty). \quad (9.2)$$

Our solution takes the form

$$\psi(x, t) = \int_{-\infty}^{\infty} G_0(x, y, t) \psi_0(y) dy, \quad (9.3)$$

where the Green function (or Feynman's propagator) is given by

$$G_0(x, y, t) = \frac{1}{\sqrt{2\pi i (\cos t \sinh t + \sin t \cosh t)}} \times \exp \left( \frac{(x^2 - y^2) \sin t \sinh t + 2xy - (x^2 + y^2) \cos t \cosh t}{2i (\cos t \sinh t + \sin t \cosh t)} \right). \quad (9.4)$$

This expression may be considered as a generalization of the propagator for the simple harmonic oscillator; see [22], [23], [24], [30], [31], [41], and references therein.

In this section, we shall extend this solution to a more general case of the forced modified oscillator with the Schrödinger equation of the form

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2} (1 + \cos 2t) \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2} (1 - \cos 2t) x^2 \psi - \frac{i}{2} \sin 2t \left( 2x \frac{\partial \psi}{\partial x} + \psi \right) - f(t) x \psi + ig(t) \frac{\partial \psi}{\partial x}, \quad (9.5)$$

where  $f(t)$  and  $g(t)$  are two arbitrary real valued functions of time only. Indeed, by a method similar to one in [38] and [41] for the case of the forced harmonic oscillator, one can look for the Green function in the form

$$\psi = u e^{iS}, \quad (9.6)$$



where  $u = G_0(x, y, t)$  is the fundamental solution of the Schrödinger equation for the modified oscillator (9.1) and  $S = \alpha(t)x + \beta(t)y + \gamma(t)$ . Its substitution into (9.5) results in

$$\begin{aligned} & \left( \frac{d\alpha}{dt}x + \frac{d\beta}{dt}y + \frac{d\gamma}{dt} \right) u \\ &= \frac{1}{2} (1 + \cos 2t) \left( -\alpha^2 u + 2i\alpha \frac{\partial u}{\partial x} \right) - \sin 2t \alpha x u + f x u + g \left( \alpha u - i \frac{\partial u}{\partial x} \right), \end{aligned} \quad (9.7)$$

where by (9.4)

$$\frac{\partial u}{\partial x} = i \frac{x (\cos t \cosh t - \sin t \sinh t) - y}{\cos t \sinh t + \sin t \cosh t} u. \quad (9.8)$$

Thus

$$\begin{aligned} & \frac{d\alpha}{dt}x + \frac{d\beta}{dt}y + \frac{d\gamma}{dt} = -\sin 2t \alpha x + f x \\ & - \frac{1}{2} (1 + \cos 2t) \left( \alpha^2 + 2\alpha \frac{x (\cos t \cosh t - \sin t \sinh t) - y}{\cos t \sinh t + \sin t \cosh t} \right) \\ & + g \left( \alpha + \frac{x (\cos t \cosh t - \sin t \sinh t) - y}{\cos t \sinh t + \sin t \cosh t} \right) \end{aligned} \quad (9.9)$$

and equating the coefficients of  $x$ ,  $y$  and 1, we obtain the following system of the first order differential equations

$$\frac{d\alpha(t)}{dt} + \frac{2}{\tan t + \tanh t} \alpha(t) = f(t) + g(t) \frac{1 - \tan t \tanh t}{\tan t + \tanh t}, \quad (9.10)$$

$$\frac{d\beta(t)}{dt} = \frac{(1 + \cos 2t) \alpha(t) - g(t)}{\cos t \sinh t + \sin t \cosh t}, \quad (9.11)$$

and

$$\frac{d\gamma(t)}{dt} = \alpha(t) g(t) - \frac{1}{2} (1 + \cos 2t) \alpha^2(t). \quad (9.12)$$

In view of

$$\frac{\mu'(t)}{\mu(t)} = \frac{2}{\tan t + \tanh t}, \quad \mu(t) = \cos t \sinh t + \sin t \cosh t, \quad (9.13)$$

equation (9.10) takes the form

$$\frac{d}{dt} (\mu(t) \alpha(t)) = \mu(t) \left( f(t) + g(t) \frac{1 - \tan t \tanh t}{\tan t + \tanh t} \right). \quad (9.14)$$

The solutions are

$$\begin{aligned} \alpha(t) &= (\cos t \sinh t + \sin t \cosh t)^{-1} \\ &\times \int_0^t (f(s) (\cos s \sinh s + \sin s \cosh s) + g(s) (\cos s \cosh s - \sin s \sinh s)) ds, \end{aligned} \quad (9.15)$$

$$\beta(t) = \int_0^t \frac{(1 + \cos 2s) \alpha(s) - g(s)}{\cos s \sinh s + \sin s \cosh s} ds, \quad (9.16)$$

$$\gamma(t) = \int_0^t \left( \alpha(s) g(s) - \frac{1}{2} (1 + \cos 2s) \alpha^2(s) \right) ds. \quad (9.17)$$

The solution of the Schrödinger equation (9.5) with the initial condition (9.2) has the form

$$\psi(x, t) = \int_{-\infty}^{\infty} G(x, y, t) \psi_0(y) dy \quad (9.18)$$

with the Green function given by

$$G(x, y, t) = G_0(x, y, t) e^{i(\alpha(t)x + \beta(t)y + \gamma(t))}. \quad (9.19)$$

Extensions for the  $n$ -dimensional case and to the corresponding heat equation are obvious. The details are left to the reader.

## 10. EXPANSION FORMULA FOR A PLANE WAVE

Equations (8.9)–(8.10) and (8.13) imply the familiar expansion formula of a plane wave in  $\mathbf{R}^n$  in terms of the hyperspherical harmonics

$$e^{i\mathbf{x} \cdot \mathbf{x}'} = rr' \left( \frac{2\pi}{rr'} \right)^{n/2} \sum_{K\nu} i^K Y_{K\nu}^*(\Omega) Y_{K\nu}(\Omega') J_{K+n/2-1}(rr'), \quad (10.1)$$

where

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} {}_0F_1 \left( \begin{matrix} - \\ \nu+1 \end{matrix} ; -\frac{(rr')^2}{4} \right) \quad (10.2)$$

is the Bessel function. Although expansion (10.1) is well known in the three dimensional case [26], we were not able to find it in the literature for the (hyper)spherical system of coordinates in  $\mathbf{R}^n$ , corresponding to a general binary tree in Vilenkin–Kuznetsov–Smorodinskiĭ’s graphical approach [46]. Dick Askey has informed us that this formula follows immediately from the Funk–Hecke theorem [1] and a familiar integral giving the Bessel functions as a Fourier transform. Bochner essentially has this in his book [17]; also see Claus Müller’s lectures on spherical harmonics [43] and [44]; it is probably in some notes of Calderon published in Argentina, but they are not widely available. Also see [10], [11], and recent papers [15] and [16].

Let us consider a few examples.

**Case  $n = 1$ .** This is simply Euler’s formula

$$e^{i\varphi} = \cos \varphi + i \sin \varphi \quad (10.3)$$

in view of the familiar relations

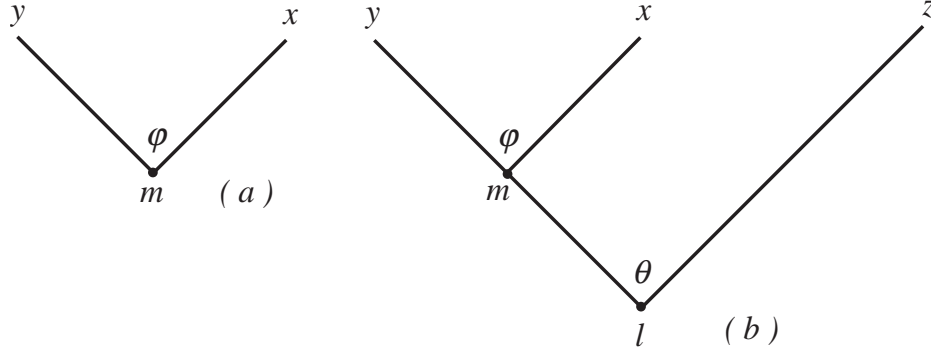
$$J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos z, \quad J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin z. \quad (10.4)$$

**Case  $n = 2$ .** Let  $x = r \cos \theta$  and  $y = r \sin \theta$ ; see Figure 2(a). Then the expansion formula (10.1) simplifies to

$$e^{irr' \cos(\theta - \theta')} = \sum_{m=-\infty}^{\infty} i^m J_m(rr') e^{im(\theta - \theta')}, \quad (10.5)$$

or

$$e^{iz \sin \varphi} = \sum_{m=-\infty}^{\infty} J_m(z) e^{im\varphi}. \quad (10.6)$$

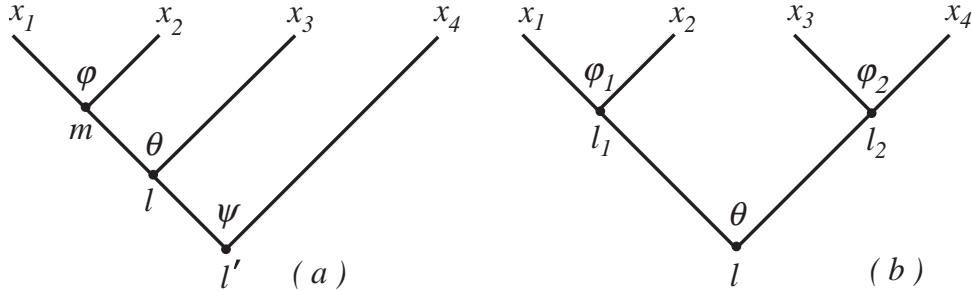
FIGURE 2. Spherical harmonics in  $\mathbf{R}^2$  and  $\mathbf{R}^3$ .

This is a well known relation in the theory of Bessel functions; see, for example, [1] and [47].

**Case  $n = 3$ .** If  $x = r \sin \theta \cos \varphi$ ,  $y = r \sin \theta \sin \varphi$ ,  $z = r \cos \theta$  (Figure 2(b)), the expansion formula (10.1) takes the familiar form

$$e^{i\mathbf{x} \cdot \mathbf{x}'} = \frac{(2\pi)^{3/2}}{\sqrt{rr'}} \sum_{l=0}^{\infty} \sum_{m=-l}^l i^l J_{l+1/2}(rr') Y_{lm}^*(\theta, \varphi) Y_{lm}(\theta', \varphi'), \quad (10.7)$$

see [26], [36], and [47] for more details.

FIGURE 3. Spherical harmonics in  $\mathbf{R}^4$ .

**Case  $n = 4$ .** Two different system of the spherical coordinates in  $\mathbf{R}^4$  are

$$\begin{aligned} x_1 &= r \sin \psi \sin \theta \sin \varphi, \\ x_2 &= r \sin \psi \sin \theta \cos \varphi, \\ x_3 &= r \sin \psi \cos \theta, \\ x_4 &= r \cos \psi \end{aligned}$$

and

$$\begin{aligned} x_1 &= r \sin \theta \sin \varphi_1, \\ x_2 &= r \sin \theta \cos \varphi_1, \\ x_3 &= r \cos \theta \sin \varphi_2, \\ x_4 &= r \cos \theta \cos \varphi_2, \end{aligned}$$

see Figures 3(a)–(b), respectively. The corresponding spherical harmonics are given by

$$Y_{l'l m}(\psi, \theta, \varphi) = A \sin^l \psi P_{l'-l}^{(l+1/2, l+1/2)}(\cos \psi) \times \sin^{|m|} \theta P_{l-|m|}^{(|m|, |m|)}(\cos \theta) e^{im\varphi} \quad (l' \geq l \geq |m|), \quad (10.8)$$

where

$$A = \frac{\sqrt{(2l+1)(2l'+2)(l-m)!(l+m)!(l'-l)!(l'+l+1)!}}{\sqrt{\pi} 2^{l+|m|-2} l! \Gamma(l'+3/2)},$$

and

$$Y_{l \ l_1 l_2}(\theta, \varphi_1, \varphi_2) = \frac{e^{i(l_1 \varphi_1 + l_2 \varphi_2)}}{2\pi} N \sin^{|l_1|} \theta \cos^{|l_2|} \theta P_{(l-|l_1|-|l_2|)/2}^{(|l_1|, |l_2|)}(\cos 2\theta) \quad (l = |l_1| + |l_2|, |l_1| + |l_2| + 2, |l_1| + |l_2| + 4, \dots) \quad (10.9)$$

with

$$N = \sqrt{\frac{(2l+2)[(l-|l_1|-|l_2|)/2]![(l+|l_1|+|l_2|)/2]!}{[(l+|l_1|-|l_2|)/2]![(l-|l_1|+|l_2|)/2]!}},$$

respectively. Here  $P_n^{(\alpha, \beta)}(\xi)$  are the Jacobi polynomials.

The expansion formulas take the forms

$$e^{i\mathbf{x} \cdot \mathbf{x}'} = \frac{(2\pi)^2}{rr'} \sum_{l'=0}^{\infty} \sum_{l=0}^{l'} \sum_{m=-l}^l i^{l'} Y_{l'l m}^*(\psi, \theta, \varphi) Y_{l'l m}^*(\psi', \theta', \varphi') J_{l'+1}(rr') \quad (10.10)$$

and

$$e^{i\mathbf{x} \cdot \mathbf{x}'} = \frac{(2\pi)^2}{rr'} \sum_{(l-|l_1|-|l_2|)/2 \geq 0} \sum_{l/2 \geq (|l_1|+|l_2|)/2 \geq 0} i^l Y_{l \ l_1 l_2}^*(\theta, \varphi_1, \varphi_2) Y_{l \ l_1 l_2}(\theta', \varphi'_1, \varphi'_2) J_{l+1}(rr'), \quad (10.11)$$

respectively.

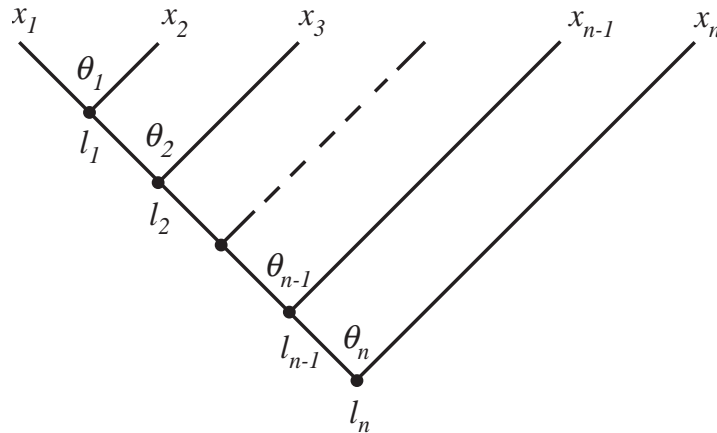


FIGURE 4. Spherical harmonics in  $\mathbf{R}^n$ .

**The  $n$ -dimensional case.** The canonical system of hyperspherical coordinates in the Euclidean space  $\mathbf{R}^n$  can be set up as follows

$$x_1 = r \sin \theta_{n-1} \sin \theta_{n-2} \cdots \sin \theta_2 \sin \theta_1,$$



with  $m = j + 1, j + 2, \dots$  similar to (3.4). When  $\delta(t) = e^{-i\omega t}$  and  $\omega = \text{constant}$ , the same considerations as in sections 3 and 6 give us the solution of the corresponding Cauchy initial value problem in the form of the expansion (11.3) with the time-dependent coefficients

$$c_m(t) = e^{-i\omega m t} \sum_{m'=j+1}^{\infty} i^{m'-m} v_{m'm}^j(2t) (\psi_{jm'}, \psi_0), \quad (11.5)$$

where  $v_{m'm}^j(2t)$  are the Bargmann functions (6.8) and

$$(\varphi, \chi) = \int_{\text{Supp } \mu} \varphi^*(x) \chi(x) d\mu(x) \quad (11.6)$$

is the inner product in the Hilbert space under consideration. A formal substitution of (11.5) into (11.3) results in

$$\psi(x, t) = \int_{\text{Supp } \mu} G(x, y, t) \psi_0(y) d\mu(y) \quad (11.7)$$

with

$$G(x, y, t) = \sum_{m=j+1}^{\infty} \sum_{m'=j+1}^{\infty} e^{-i\omega m t} i^{m'-m} v_{m'm}^j(2t) \psi_{jm}(x) \psi_{jm'}(y). \quad (11.8)$$

The original Cauchy initial value problem (1.7)–(1.9) provides an explicit model of the abstract Hilbert space. Another realization is a quasipotential model of the relativistic oscillator; see, for example, [37], [7], and [8]. We shall discuss this model and its generalization in the next two sections. Propagators for the difference models of the simple harmonic oscillator [9] can be derived in a similar fashion. The details are left to the reader.

## 12. PROPAGATOR FOR THE RELATIVISTIC OSCILLATOR

For the consistent three-dimensional description of a relativistic two-particle system in quantum field theory a quasipotential approach had been formulated and, in the framework of this approach, some relativistic generalizations of the exactly solvable problems of quantum mechanics have been considered; see [37], [7], and references therein for more information. We shall discuss here a model of the (simple) relativistic oscillator described by the following Hamiltonian operator

$$H_0 = mc^2 \cosh(i\lambda \partial_x) + \frac{1}{2} m\omega^2 x(x + i\lambda) \exp(i\lambda \partial_x), \quad (12.1)$$

where  $\lambda = \hbar/mc$  is the Compton wave length,  $\partial_x = \partial/\partial x$  and  $\exp(\alpha \partial_x) f(x) = f(x + \alpha)$  is the shift operator. The square-integrable solutions of the stationary Schrödinger equation

$$H_0 \Psi_n(x) = E_n \Psi_n(x) \quad (-\infty < x < \infty) \quad (12.2)$$

on the real line, which correspond to the discrete energy levels

$$E_n = \hbar\omega (n + \nu), \quad \nu = \frac{1}{2} + \sqrt{\frac{1}{4} + \left(\frac{c}{\lambda\omega}\right)^2} \quad (n = 0, 1, 2, \dots), \quad (12.3)$$

can be found in terms of the special Meixner–Pollaczek polynomials [8] as follows

$$\Psi_n(x) = 2^\nu \sqrt{\frac{n!}{2\pi\lambda\Gamma(n+2\nu)}} (\nu(\nu-1))^{-ix/2\lambda} \Gamma(\nu + ix/\lambda) P_n^\nu(x/\lambda, \pi/2). \quad (12.4)$$

On the other hand, solution of the Cauchy initial value problem for the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = H_0 \psi \quad (12.5)$$

with the Hamiltonian (12.1) subject to the initial condition

$$\psi(x, t)|_{t=0} = \psi_0(x) \in \mathcal{L}^2(-\infty, \infty) \quad (12.6)$$

has the form

$$\begin{aligned} \psi(x, t) &= \sum_{n=0}^{\infty} e^{-i(E_n t)/\hbar} \Psi_n(x) \int_{-\infty}^{\infty} \Psi_n^*(y) \psi_0(y) dy \\ &= \int_{-\infty}^{\infty} G_0(x, y, t) \psi_0(y) dy, \end{aligned} \quad (12.7)$$

where the Green function (or Feynman's propagator)

$$G_0(x, y, t) = \sum_{n=0}^{\infty} e^{-i\omega(n+\nu)t} \Psi_n(x) \Psi_n^*(y) \quad (12.8)$$

can be found in a closed form as follows

$$\begin{aligned} G_0(x, y, t) &= \frac{1}{2\pi\lambda} \left( \frac{c}{\lambda\omega} \right)^{i(y-x)/\lambda} (i \sin(\omega t/2))^{i(y-x)/\lambda} (\cos(\omega t/2))^{-i(x+y)/\lambda} \\ &\quad \times \Gamma(i(x-y)/\lambda) {}_2F_1 \left( \begin{matrix} \nu - ix/\lambda, 1 - \nu - ix/\lambda \\ 1 + i(y-x)/\lambda \end{matrix}; \sin^2(\omega t/2) \right) \\ &+ \frac{1}{2\pi\lambda} \left( \frac{c}{\lambda\omega} \right)^{i(y-x)/\lambda} (i \sin(\omega t/2))^{i(x-y)/\lambda} (\cos(\omega t/2))^{-i(x+y)/\lambda} \\ &\quad \times \Gamma(i(y-x)/\lambda) \frac{\Gamma(\nu + ix/\lambda) \Gamma(\nu - iy/\lambda)}{\Gamma(\nu - ix/\lambda) \Gamma(\nu + iy/\lambda)} \\ &\quad \times {}_2F_1 \left( \begin{matrix} \nu - iy/\lambda, 1 - \nu - iy/\lambda \\ 1 + i(x-y)/\lambda \end{matrix}; \sin^2(\omega t/2) \right). \end{aligned} \quad (12.9)$$

We have used here the Poisson kernel for the Meixner–Pollaczek polynomials (4.11) in order to sum the series and a transformation formula for the analytic continuation of the hypergeometric function [1] and [47].

### 13. A MODIFIED RELATIVISTIC OSCILLATOR

As is known [7] a dynamical symmetry group for the relativistic oscillator with the Hamiltonian (12.1) is the group  $SU(1, 1)$  (or isomorphic groups  $SO(2, 1) \sim Sp(2, \mathbf{R}) \sim SL(2, \mathbf{R})$ ), whose generators are realized as the difference operators

$$K_0 = \frac{1}{\hbar\omega} H_0, \quad K_{\pm} = \frac{x}{\lambda} \pm iK_0 \mp \frac{ic}{\lambda\omega} \exp(-i\lambda\partial_x) \quad (13.1)$$

acting on the eigenfunctions of the Hamiltonian (12.1) as follows

$$\begin{aligned} K_+ \Psi_n(x) &= \sqrt{(n+1)(n+2\nu)} \Psi_{n+1}(x), \\ K_- \Psi_n(x) &= \sqrt{n(n+2\nu-1)} \Psi_{n-1}(x), \\ K_0 \Psi_n(x) &= (n+\nu) \Psi_n(x). \end{aligned} \quad (13.2)$$

The wavefunctions  $\Psi_n(x) = \psi_{jm}$  with  $n = m - j - 1$  and  $\nu = j + 1$  form the basis for the infinite-dimensional irreducible unitary representations of the discrete positive series  $\mathcal{D}_+^j$  of the universal covering group  $\widetilde{SU}(1, 1)$ ; see equations (2.19).

In this section, we shall solve the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = H(t) \psi \quad (13.3)$$

with a modified Hamiltonian of the form

$$H(t) = H_0 + \hbar(\delta(t) K_+ + \delta^*(t) K_-), \quad \delta(t) = e^{-i\omega t} \quad (13.4)$$

subject to the initial condition (12.6). The eigenfunction expansion

$$\psi(x, t) = \sum_{m=j+1}^{\infty} c_m(t) \psi_{jm}(x) \quad (13.5)$$

leads to the familiar system (3.4), whose solutions in terms of the Bargmann functions are

$$c_m(t) = e^{-i\omega m t} \sum_{m'=j+1}^{\infty} i^{m'-m} v_{m'm}^j(2t) \int_{-\infty}^{\infty} \psi_{jm'}^*(y) \psi_0(y) dy. \quad (13.6)$$

Thus

$$\psi(x, t) = \int_{-\infty}^{\infty} G(x, y, t) \psi_0(y) dy, \quad (13.7)$$

where the Green function is given by

$$G(x, y, t) = \sum_{m=j+1}^{\infty} \sum_{m'=j+1}^{\infty} e^{-i\omega m t} i^{m'-m} v_{m'm}^j(2t) \psi_{jm}(x) \psi_{jm'}^*(y). \quad (13.8)$$

This multiple series can be simplified to two single sums. Indeed, rewriting the wave functions (12.4) in terms of the hypergeometric function

$$\begin{aligned} \psi_{jm}(x) &= \frac{2^{j+1}}{\sqrt{2\pi\lambda}} \frac{(j(j+1))^{-ix/2\lambda} \Gamma(j+1+ix/\lambda)}{\Gamma(2j+2)} \sqrt{\frac{(m+j)!}{(m-j-1)!}} \\ &\quad \times (-i)^{m-j-1} {}_2F_1\left(\begin{matrix} -m+j+1, j+1-ix/\lambda \\ 2j+2 \end{matrix}; 2\right) \end{aligned} \quad (13.9)$$

by (4.7) and (4.1) and then using the Meixner generating relation (4.3), we have

$$\begin{aligned} &\sum_{m'=j+1}^{\infty} i^{m'-j-1} v_{m'm}^j(2t) \psi_{jm'}^*(y) \\ &= \frac{2^{j+1}}{\sqrt{2\pi\lambda}} \frac{(j(j+1))^{iy/\lambda} \Gamma(j+1-iy/\lambda)}{\Gamma(2j+2)} e^{-2ity/\lambda} \\ &\quad \times (-1)^{m-j-1} \sqrt{\frac{(m+j)!}{(m-j-1)!}} {}_2F_1\left(\begin{matrix} -m+j+1, j+1+iy/\lambda \\ 2j+2 \end{matrix}; 2\right) \\ &= i^{m-j-1} e^{-2ity/\lambda} \psi_{jm}^*(y). \end{aligned} \quad (13.10)$$



Thus, say for a pure imaginary time,

$$\begin{aligned}
G(x, y, t) &= \sum_{m=j+1}^{\infty} e^{-i\omega m t} (-i)^{m-j-1} \psi_{jm}(x) \\
&\times \sum_{m'=j+1}^{\infty} i^{m'-j-1} v_{m'm}^j(2t) \psi_{jm'}^*(y) \\
&= \frac{1}{2\pi\lambda} (j(j+1))^{i(y-x)/2\lambda} \frac{\Gamma(j+1+ix/\lambda) \Gamma(j+1-iy/\lambda)}{\Gamma(2j+2)} \\
&\times e^{-2ity/\lambda} (\cos(\omega t/2))^{-2j-2} (i \tan(\omega t/2))^{i(y-x)/\lambda} \\
&\times {}_2F_1 \left( \begin{matrix} j+1-ix/\lambda, j+1+iy/\lambda \\ 2j+2 \end{matrix} ; \frac{1}{\cos^2(\omega t/2)} \right).
\end{aligned} \tag{13.11}$$

Here we use a transformation formula for the analytic continuation of the hypergeometric function [1] and [47] in order to obtain the final result as follows

$$\begin{aligned}
G(x, y, t) &= \frac{1}{2\pi\lambda} \left( \frac{c}{\lambda\omega} \right)^{i(y-x)/\lambda} e^{-2ity/\lambda} (i \sin(\omega t/2))^{i(y-x)/\lambda} (\cos(\omega t/2))^{-i(x+y)/\lambda} \\
&\times \Gamma(i(x-y)/\lambda) {}_2F_1 \left( \begin{matrix} \nu-ix/\lambda, 1-\nu-ix/\lambda \\ 1+i(y-x)/\lambda \end{matrix} ; \sin^2(\omega t/2) \right) \\
&+ \frac{1}{2\pi\lambda} \left( \frac{c}{\lambda\omega} \right)^{i(y-x)/\lambda} e^{-2ity/\lambda} (i \sin(\omega t/2))^{i(x-y)/\lambda} (\cos(\omega t/2))^{i(x+y)/\lambda} \\
&\times \Gamma(i(y-x)/\lambda) \frac{\Gamma(\nu+ix/\lambda) \Gamma(\nu-iy/\lambda)}{\Gamma(\nu-ix/\lambda) \Gamma(\nu+iy/\lambda)} \\
&\times {}_2F_1 \left( \begin{matrix} \nu+ix/\lambda, 1-\nu+ix/\lambda \\ 1+i(x-y)/\lambda \end{matrix} ; \sin^2(\omega t/2) \right).
\end{aligned} \tag{13.12}$$

Comparing expressions (12.9) and (13.12) for two propagators, one can see that

$$G(x, y, t) = G_0(x, y, t) e^{-2ity/\lambda}. \tag{13.13}$$

It also follows from (12.8), (13.9), (13.10), and (13.11). The details are left to the reader.

#### 14. A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

Consider the following infinite system of ordinary differential equations

$$i \frac{du_n(t)}{dt} = (n+1) u_{n+1}(t) - 2(n+\lambda) \cos \varphi u_n(t) + (n+2\lambda-1) u_{n-1}(t), \tag{14.1}$$

with  $u_{-1}(t) = 0$  ( $n = 0, 1, 2, \dots$ ) subject to the initial conditions

$$u_n(0) = u_n^0. \tag{14.2}$$

Looking for a particular solution in the form

$$u_n(t) = e^{-2ixt \sin \varphi} P_n^\lambda(x, \varphi), \tag{14.3}$$

one gets the three term recurrence relation (4.8) for the Meixner–Pollaczek polynomials  $P_n^\lambda(x, \varphi)$ . The particular solution that satisfies the initial conditions  $v_{nm}(0) = \delta_{nm}$  is given in the integral form

$$u_{nm}(t) = \frac{1}{d_m^2} \int_{-\infty}^{\infty} e^{-2ixt \sin \varphi} P_n^\lambda(x, \varphi) P_m^\lambda(x, \varphi) \rho(x) dx, \quad (14.4)$$

where  $\rho(x)$  and  $d_m^2$  are the weight function and the squared norm for the Meixner–Pollaczek polynomials  $P_n^\lambda(x, \varphi)$ , respectively; see (4.9)–(4.10). The solution of the initial value problem is

$$u_n(t) = \sum_{m=0}^{\infty} u_m^0 u_{nm}(t). \quad (14.5)$$

The last integral can be evaluated as a single sum with the help of Meixner’s generating relation (4.3). The final result is

$$\begin{aligned} u_{nm}(t) &= \frac{(2\lambda)_n}{n!} \frac{e^{i\pi\lambda} (\sin \varphi)^{2\lambda} (\sinh(t \sin \varphi))^{n+m}}{(\cos \varphi \sinh(t \sin \varphi) + i \sin \varphi \cosh(t \sin \varphi))^{n+m+2\lambda}} \\ &\quad \times {}_2F_1 \left( \begin{matrix} -n, -m \\ 2\lambda \end{matrix} ; - \left( \frac{\sin \varphi}{\sinh(t \sin \varphi)} \right)^2 \right). \end{aligned} \quad (14.6)$$

This is an extension of (6.3)–(6.6) where  $\varphi = \pi/2$ .

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## 15. APPENDIX. ANOTHER INTEGRAL EVALUATION

The following integral

$$\begin{aligned} &\int_0^\infty e^{-\lambda z} z^{\gamma-1} {}_1F_1 \left( \begin{matrix} \alpha \\ \gamma \end{matrix} ; kz \right) {}_1F_1 \left( \begin{matrix} \alpha' \\ \gamma \end{matrix} ; k'z \right) dz \\ &= \Gamma(\gamma) \lambda^{\alpha+\alpha'-\gamma} (\lambda - k)^{-\alpha} (\lambda - k')^{-\alpha'} {}_2F_1 \left( \begin{matrix} \alpha, \alpha' \\ \gamma \end{matrix} ; \frac{kk'}{(\lambda - k)(\lambda - k')} \right) \end{aligned} \quad (15.1)$$

is evaluated in [29] and [36]. When  $\alpha' = \alpha$ , replace  $\lambda = \mu\alpha$ ,  $z = x/\alpha$  and take the limit  $\alpha = n \rightarrow \infty$  with the help of

$$\lim_{\alpha \rightarrow \infty} {}_1F_1 \left( \begin{matrix} \alpha \\ \gamma \end{matrix} ; \frac{kx}{\alpha} \right) = {}_0F_1 \left( \begin{matrix} - \\ \gamma \end{matrix} ; kx \right), \quad (15.2)$$

$$\lim_{\alpha \rightarrow \infty} {}_2F_1 \left( \begin{matrix} \alpha, \alpha \\ \gamma \end{matrix} ; \frac{kk'}{(\mu\alpha - k)(\mu\alpha - k')} \right) = {}_0F_1 \left( \begin{matrix} - \\ \gamma \end{matrix} ; \frac{kk'}{\mu^2} \right), \quad (15.3)$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{\beta}{n}\right)^n = e^\beta. \quad (15.4)$$

The end result

$$\begin{aligned} & \int_0^\infty e^{-\mu x} x^{\gamma-1} {}_0F_1\left(\begin{matrix} - \\ \gamma \end{matrix}; kx\right) {}_0F_1\left(\begin{matrix} - \\ \gamma \end{matrix}; k'x\right) dx \\ &= \Gamma(\gamma) \mu^{-\gamma} e^{(k+k')/\mu} {}_0F_1\left(\begin{matrix} - \\ \gamma \end{matrix}; \frac{kk'}{\mu^2}\right) \end{aligned} \quad (15.5)$$

is required in section 8 of this paper.

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